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THE STABILITY OF PERIODIC HAMILTONIAN SYSTEMS WITH MULTIPLE FOURTH-ORDER RESONANCE†

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The problem of the stability of the periodic motion of a non-linear periodic Hamiltonian system is considered in the case of pure imaginary characteristic exponents which also satisfy several fourth-order resonance conditions. Conditions for stability and instability are formulated based on terms of the third order inclusive. Some conclusions generalize results obtained previously [1]. © 1999 Elsevier Science Ltd. All rights reserved.

Consider the problem of the stability of a stationary point of a Hamiltonian system of equations with Hamiltonian $H(x, y, t) = H_2 + H_3 + \ldots$, where H_l are the lth order forms in the variables $x = (x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_N)$ with coefficients which are periodic in t with period ω. We will assume that all characteristic exponents $\lambda_s = i\omega_s$ of the linear part of the corresponding canonical system of equations are pure imaginary and different, and in addition satisfy several simultaneous relationships of internal fourth-order resonance

$$\langle \mathbf{p}, \Omega \rangle = 2\pi\omega^{-1}q; \quad q = 0, \pm1, \pm2, \ldots \tag{1}$$

and there are no resonances of order less than four. Here $\mathbf{p}$ is an n-vector whose components are relatively prime integers, not all simultaneously zero, $|p_1| +, \ldots, +|p_n| = 4$ and $\Omega = |\omega_1, \ldots, \omega_n|$ is the vector of characteristic exponents $(1 \leqslant n \leqslant N)$.

The case of double resonance of the form (1) for autonomous systems was considered previously in [1], where sufficient conditions were obtained for a model system (i.e. a system containing terms of order up to and including three) to be stable.

The aim of this paper is to extend those results to periodic systems and to consider new types of resonance that arise only in periodic systems. Throughout what follows it will be assumed that the form H_2 has already been reduced to canonical form $2H_2 = \omega_1(x_1^2 + y_1^2) + \ldots + \omega_N(x_N^2 + y_N^2)$; as we know [2], this is always possible.

We will first consider the case in which all resonances (1) are independent (that is, do not contain common exponents). Suppose there are μ resonance relations, each containing n_ν $(\nu = 1, \ldots, \mu)$ exponents, so that

$$\langle p_\nu, \Omega_\nu \rangle = 2\pi\omega^{-1}q_\nu; \quad q_\nu = 0, \pm1, \pm2, \ldots \quad (\nu = 1, \ldots, \mu) \tag{2}$$

where $\mathbf{p}_\nu = |p_{m_{\nu-1}+1}, \ldots, |$, $\Omega_\nu = |\omega_{m_{\nu-1}+1}, \ldots, \omega_{m_\nu}|$; in these relations, $n_0 = 0$

$$n_1 + \ldots + n_\mu = n \leqslant N, \quad m_\nu = n_1 + \ldots + n_\nu, \text{ a } \|\mathbf{p}_\nu\| = |p_{m_{\nu-1}+1}| +, \ldots, +|p_{m_\nu}| = 4$$

We know [3–5] that in the case of a single resonance (1) the initial system may be reduced, by a polynomial canonical transformation with periodic coefficients, to normal form in which there are no terms of even order, while the terms of odd order do not contain the time t. The same result may be achieved in the case of multiple resonance (2). In specially chosen polar coordinates r_j, θ_j [5], the normalized part (up to terms of the fourth order inclusive) of the Hamiltonian takes the form

$$2H = \sum_{j=1}^{N} \omega_j r_j + 2H_4 \tag{3}$$

$$H_4 = \sum_{\nu=1}^{\mu} A_\nu \sqrt{R_\nu} \cos\psi_\nu + \sum_{i,j=1}^{N} A_{ij} r_i r_j, \qquad R_\nu = \Pi r_\alpha^{p_\alpha}, \quad \psi_\nu = \sum p_\alpha \theta_\alpha$$

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where the subscript α in the sums and products takes all values from $m_{\nu-1}+1$ to m_ν, where $m_{\nu-1} = n_1 + \ldots + n_{\nu-1}$.

The normal form obtained here is identical with that presented previously [1] for $\mu = 2$. In addition, representation (3) does not exclude the presence of "single-frequency" resonances (2) (i.e. corresponding to $n_\nu = 1$), which are impossible in the autonomous case.

As in [1], we will call a resonance weak if, in the absence of other resonances, it does not cause the model system to be unstable; otherwise, we will call it strong.

It is easy to prove the following theorem for the trivial solution of the model system corresponding to Hamiltonian (3).

Theorem 1. If at least one of resonances (2) is strong, the trivial solution of the model system is unstable.

Suppose the resonance with $\nu = \gamma$ in (2) is strong. The, equating all r_j in the canonical system for the Hamiltonian (3) to zero, except those for which $j = m_{\gamma-1}+1, \ldots, m_\gamma$, we obtain a system of equations describing a situation with one resonance, for which, by assumption, the trivial solution is unstable. Hence it follows that the trivial solution of the entire initial system is also unstable.

The case in which all resonances are weak is more complicated. One must then make a distinction between two types of weak resonance: A—weakness of a resonance due to changes of sign among the components of the resonance vector $\mathbf{p}_\nu$ (in that case, stability in any finite order [3]), and B—all components of the resonance vector $\mathbf{p}_\nu$ are of the same sign, and the weakness of each resonance is due to the inequalities

$$|A_\nu| < |S_\nu| \quad (\nu = 1, \ldots, \mu); \qquad S_\nu = \left(\sum A_{\alpha\beta} p_\alpha p_\beta\right) \Big/ \left(2\sqrt{P_\nu}\right), \quad P_\nu = \Pi p_\alpha^{p_\alpha}$$

The subscripts α and β in the sums and products take all values from $m_{\nu-1}+1$ to m_ν, where $m_{\nu-1} = n_1 + \ldots + n_{\nu-1}$.

Theorem 2. Suppose there is no single-frequency resonance in the system and all resonances are weak in sense A. Then the trivial solution of the model system corresponding to the Hamiltonian (3) is stable.

In that case the model system corresponding to Hamiltonian (3) has a sign definite integral

$$\Phi = \sum_{i=1}^{N} \gamma_i r_i = \text{const}, \quad \gamma_i = \text{const} > 0$$

Indeed, the requirement that the derivative $\Phi^{\cdot}$ should vanish identically along trajectories of the model system implies the identity

$$\dot{\Phi} = -2 \sum_{\nu=1}^{\mu} A_\nu \sqrt{R_\nu} \sin \psi_\nu \left(\sum \gamma_\alpha p_\alpha\right) \equiv 0$$

which may hold only provided that $\Sigma \gamma_\alpha p_\alpha = 0$ (the subscript α takes all values from $m_{\nu-1}+1$ to m_ν, $\nu = 1, \ldots, \mu$).

The equations obtained for γ_α always have a strictly positive solution if there is a change of sign among the numbers p_α, and so Φ is indeed a positive definite integral, proving that the model system is stable.

The case of weak resonances in sense B is more complicated.

Theorem 3. Suppose all μ resonances in the system are weak and some of them are weak in sense B. Then the trivial solution of the model system corresponding to the Hamiltonian (3) is stable if there are no changes of sign among the quantities S_i $(i = 1, \ldots, m)$ and $q_i = 0$.

Without loss of generality, we will assume that the first k resonances are weak in sense A and all the others in sense B. In that case the system has the following integrals

$$\Phi = \sum_{j=1}^{l} \gamma_j r_j + \sum_{i=n+1}^{N} \gamma_i r_i \quad (l = n_1 + \ldots + n_k)$$

$$I_s = r_s - \frac{p_s}{p_{m_{\nu-1}+1}} r_{m_{\nu-1}+1} \quad (s = m_{\nu-1}+2, \ldots, m_\nu,\ \nu = k+1, \ldots, \mu) \tag{4}$$

$$H_4 = \sum_{\nu=1}^{\mu} A_\nu \sqrt{R_\nu} \cos \psi_\nu + \sum_{i,j=1}^{N} A_{ij} r_i r_j$$

from which we construct another integral (summation over s is performed according to (4))

$$G = \Phi^4 + \sum I_s^4 + H_4^2$$

which is sign definite. Indeed, for $r_j = r_i = 0$, $r_s = p_j/p_{m_{\nu-1}+1} r_{n_{\nu-1}+1}$, we have

$$\Phi = I_s = 0, \quad H_4 = 2\sum_{\nu=k+1}^{\mu} \frac{r_{m_{\nu-1}+1}^2}{p_{m_{\nu-1}+1}^2}\sqrt{P_\nu}(A_\nu \cos\theta_\nu + S_\nu)$$

and hence, taking into account that the resonances are weak in sense B and that there are no changes of sign among $S_\nu(\nu = k+1, \ldots, \mu)$, we verify that G is a positive definite function.

Corollary 1. Suppose all μ resonances in the system are weak and only one of them is weak in sense B. Then the trivial solution of the model system corresponding to Hamiltonian (3) is stable.

We will now consider the case of the interaction of resonances associated with one frequency. Suppose there are μ resonance relations, each containing $n_\nu(\nu = 1, \ldots, \mu)$ exponents, so that

$$p_\nu^*\omega_0 + \langle \mathbf{p}_\nu, \Omega_\nu \rangle = 2\pi\omega^{-1} q_\nu; \quad q_\nu = 0, \pm 1, \pm 2, \ldots \; (\nu = 1, \ldots, \mu) \tag{5}$$

where $\mathbf{p}_\nu = |p_{m_{\nu-1}+1}, \ldots, p_{m_\nu}|$, $\Omega_\nu = |\omega_{m_{\nu-1}+1}, \ldots \omega_{m_\nu}|$; it is assumed here that $n_0 = 0$, $n_1 + \ldots + n_\mu = n \leqslant N$, $m_\nu = n_1 + \ldots + n_\nu$ and $|p_\nu^*| + \|\mathbf{p}_\nu\| = 4$.

In specially chosen polar coordinates r_j, θ_j [5], the normalized part (up to terms of fourth order inclusive) of the Hamiltonian takes the same form as (3)

$$2H = \sum_{j=1}^{N} \omega_j r_j + 2H_4 \tag{6}$$

$$H_4 = \sum_{\nu=1}^{\mu} A_\nu \sqrt{R_\nu}\cos\psi_\nu + \sum_{i,j=1}^{N} A_{ij} r_i r_j, \quad R_\nu = r_0^{|p_\nu^*|}\Pi r_\alpha^{|p_\alpha|}, \quad \psi_\nu = p_\nu^*\theta_0 + \sum p_\alpha\theta_\alpha$$

This normal form is identical with that presented previously [1] for $\mu = 2$.

It can be verified that Theorem 1 holds for the trivial solution of a model system corresponding to Hamiltonian (6).

If there is interaction of weak resonances that contain one common frequency, the situation turns out to be more complicated than in the case of independent resonances. In that case the sufficient conditions for stability yield the following theorems.

Theorem 4. Suppose all resonances are weak in sense A. Then the trivial solution of the model system corresponding to Hamiltonian (6) is stable.

The proof is again based on the existence of a sign definite integral

$$\Phi = \sum_{i=0}^{N} \gamma_i r_i = \text{const}$$

where γ_i are positive constants. Indeed, the requirement that the derivative $\Phi^\cdot$ should vanish identically implies the identity

$$\dot{\Phi} = -2\sum_{\nu=1}^{\mu} A_\nu \sqrt{R_\nu} \sin\Psi_\nu\left(\sum \gamma_\alpha p_a + \gamma_0 p_\nu^*\right) \equiv 0$$

which may hold only provided that $\Sigma\, \gamma_\alpha p_a + \gamma_0 p_\nu^* = 0$ (the subscript α takes all values from $m_{\nu-1} + 1$ to $m_{\nu-1} + m_\nu$, $\nu = 1, \ldots, \mu$).

These equations for γ_α always have a strictly positive solution if there is a change of sign among the numbers p_α, p_ν^*, and therefore Φ is indeed a positive definite integral, proving that the model system is stable.

The case in which the resonances are weak in sense B is more complicated.

Let

$$S_{\gamma\nu} = \sum_i\sum_j C_{ij} p_i p_j \quad (j = m_{\gamma-1} + 1, \ldots, m_\gamma,\; i = m_{\nu-1} + 1, \ldots, m_\nu)$$

Theorem 5. Suppose all μ resonances in the system are weak and some of them are weak in sense B. Then the trivial solution of the model system corresponding to Hamiltonian (6) is stable if there is no change of sign among the quantities S_i, S_{ij} ($i = 1, \ldots, \mu - k, j = 1, \ldots, \mu - k$), but there is a change of sign among the components of the resonance vector $\mathbf{p}_\nu$ corresponding to weak resonance in sense A, and $q_i = 0$.

Without loss of generality, we will assume that the first k resonances are weak in sense A and the others in sense B. The system will have the following integrals

$$\begin{aligned}
&\Phi = \sum_{s=1}^{l} \gamma_s r_s + \sum_{i=n+1}^{N} \gamma_i r_i \quad (l = n_1 + \ldots + n_k) \\
&I_j = r_j - \frac{p_j}{p_{m_{\nu-1}+1}} r_{m_{\nu-1}+1} \quad (j = m_{\gamma-1} + 2, \ldots, m_\gamma, \nu = k+1, \ldots, \mu) \\
&\hat{I}_{sj} = r_0 - \frac{p_s}{p_{m_{\upsilon-1}+1}} r_{m_{\upsilon-1}+1} - \frac{p_j}{p_{m_{\nu-1}+1}} r_{m_{\nu-1}+1} \quad (\upsilon = 1, \ldots, k, \nu = k+1, \ldots, \mu) \\
&H_4 = \sum_{\nu=1}^{\mu} A_\nu \sqrt{R_\nu} \cos\psi_\nu + \sum_{\alpha,\beta=1}^{N} A_{\alpha\beta} r_\alpha r_\beta
\end{aligned} \tag{7}$$

from which we construct the following integral (summation over the subscripts s and j is performed in accordance with (7))

$$G = \Phi^4 + \sum I_j^4 + \sum\sum I_{sj}^4 + H_4^2$$

which is sign definite. Indeed, if

$$r_s = r_i = 0, \quad r_j = p_j / p_{m_{\nu-1}+1} r_{m_{\nu-1}+1}, \quad r_0 = p_j^* / p_{n_{\nu-1}+1} r_{n_{\nu-1}+1}$$

we have

$$\Phi = \hat{I}_{sj} = I_j = 0$$

$$H_4 = \sum_{\upsilon=k+1}^{\mu} \left(2\frac{r^2_{m_{\upsilon-1}+1}}{p^2_{m_{\upsilon-1}+1}} \sqrt{P_\upsilon} (A_\upsilon \cos\theta_\upsilon + S_\upsilon) + \sum_{\substack{\nu=k+1 \\ (\nu\neq\upsilon)}}^{\mu} \frac{r_{m_{\upsilon-1}+1} r_{m_{\nu-1}+1}}{p_{m_{\upsilon-1}+1} p_{m_{\nu-1}+1}} S_{\nu\upsilon} \right)$$

and hence, taking into account that the resonances are weak in sense B and that there are no changes of sign among S_υ, $S_{\nu\upsilon}$ ($\upsilon = k + 1, \ldots, \mu, \nu = k + 1, \ldots, \mu$), we verify that G is a positive definite function.

Corollary 2. Suppose all μ resonances in the system are weak in sense B. Then the trivial solution of the model system corresponding to Hamiltonian (6) is stable if there is no change of sign among the quantities S_i, S_{ij} ($i = 1, \ldots, n, j = 1, \ldots, n$) and $q_i = 0$.

Consider the case in which the common component of a multiple resonance is a single-frequency resonance. Then the following theorem is true.

Theorem 6. If a single-frequency resonance is weak, all other resonances are weak in sense B and there is a change of sign among the components of the resonance vector $\mathbf{p}_\nu$ corresponding to weak resonance in sense A, then the trivial solution of the model system is stable.

Note that the system has integrals

$$\begin{aligned}
&\Phi = \sum_{s=1}^{l} \gamma_s r_s + \sum_{i=n+1}^{N} \gamma_i r_i \quad (l = n_1 + \ldots + n_k) \\
&I_j = r_j - r_{00} - \frac{p_j}{p_{m_{\nu-1}+1}} r_{m_{\nu-1}+1} \\
&(j = m_{\gamma-1} + 2, \ldots, m_\gamma, \nu = k+1, \ldots, \mu, \dot{r}_{00} = -4A_0 r_0^2 \cos(4\theta_0 - \varphi_0)) \\
&H_4 = \sum^{\mu} A_\nu \sqrt{R_\nu} \cos\psi_\nu + \sum^{N} A_{\alpha\beta} r_\alpha r_\beta + A_0 r_0^2 \sin(4\theta_0 - \varphi_0)
\end{aligned} \tag{8}$$

from which we can construct the following integral (summation over the subscript j is performed in accordance with (8))

$$G = \Phi^4 + \sum I_j^4 + H_4^2$$

which is sign definite. Indeed, if $r_s = r_i = 0$, $r_0 = r_{00}$, we have

$$\Phi = I_j = 0, \quad H_4 = r_{00}^2(A_0 \sin(4\theta_0 - \varphi_0) + C_{00})$$

and hence, taking into account that the single-frequency resonance is weak (i.e. $C_{00} > A_0$), we verify that G is a positive definite function.

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